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Declining valuations and equilibrium bidding in central bank refinancing operations ☆

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ABSTRACT

Among the most puzzling observations on the euro money market are the discount in the weekly refinancing operations, the more aggressive bidding under uncertainty, the temporary flatness of bid schedules, and the development of interest rate spreads. To explain these observations, we consider a standard divisible-good auction with either uniform or discriminatory pricing, and place it in the context of a secondary market for interbank credit. The analysis links the empirical evidence to the endogenous choice of collateral in credit transactions. We also discuss the Eurosystem's preference for the discriminatory auction, the remuneration of reserves, and the impact of the recent market turmoil.

1. Introduction

The euro money market, defined here as the market for euro-denominated short-term credit between counterparties of the Eurosystem, has been challenging economists by exhibiting a variety of puzzling features right from the market's inception in January 1999.[1] One of these features has been that credit seems to be obtainable at more attractive conditions in the primary market, i.e., in central bank operations, than in the secondary market, i.e., in the interbank market. This is counterintuitive because the regular central bank operations in the euro area, the so-called main refinancing operations, are effectively highest-price auctions involving several hundred bidders. How is a discount possible under such tight competition?[2]

The first account and explanation of the discount has been given in a seminal contribution by Ayuso and Repullo (2003) who assume that the central bank has an *asymmetric objective function* and penalizes downwards deviations of the market rate more heavily than upwards deviations. As a consequence, the central bank follows a tight allotment policy, driving the market rate above the policy rate, which explains excessive bidding for central bank reserves. However, the explanation depends on the central bank's use of the so-called fixed-rate tender, through which funds are offered to market participants below market conditions. The model, therefore, cannot account for the spread between primary and secondary market conditions under the variable-tender regime which has been in place following June 2000.

Another potential explanation of the spread between primary and secondary market conditions might be *intermediation* within the banking sector.[3] Indeed, Freixas and Holthausen (2004) have pointed out the role of money centers to distribute unexpected liquidity shocks within the euro area. Intermediation is useful in this context because it reduces the informational frictions of unsecured lending, especially in a cross-border context. However, in contrast to the case of unsecured

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[1] Background information on the Eurosystem auctions and the euro money market is provided in Appendix A.

[2] See Section 2 for an overview over the empirical evidence.

[3] Neyer and Wiemers (2004) have developed a model along these lines.

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interbank lending considered by Freixas and Holthausen, all loans vis-à-vis the Eurosystem have to be fully collateralized. This suggests that informational frictions are not a central feature of lending allotted through Eurosystem tenders.[4] To the contrary, such intermediation is likely to cause unnecessary costs, for instance in terms of regulatory capital usage,[5] counterparty risk, or collateral handling. In sum, this suggests that we should find actually only limited intermediation, at least outside of banking groups, of refinancing that had been received in the Eurosystem tenders.[6] We conclude that intermediation is not likely to fully explain the spread in money market conditions.[7]

To better understand this and other pieces of evidence, the present paper offers a theoretical framework that integrates two institutional features of the Eurosystem's variable-rate tenders. The first element, adopted from Klemperer and Meyer's (1989) analysis of oligopolistic competition, is an *aggregate uncertainty* (potentially small, but not negligible) about the quantity that is eventually allotted in the auction.[8] The second element is an *endogenous choice of collateral* pledged to secure the individual funding transaction. The framework is then used to study equilibrium bidding of commercial banks in Eurosystem auctions in the context of a competitive secondary market, where we allow both the uniform and the discriminatory pricing rule.[9]

The characterization of the equilibrium is shown to have a number of testable implications, which are compared to the empirical evidence. First, we look at the discount between primary and secondary market conditions, as discussed above. The model predicts here that conditions offered through the discriminatory auction are typically strictly below marginal valuations *even if there are many bidders*. This suggests an explanation for the obscure underpricing. Second, empirical research tells us that with more uncertainty in the market, bids are on average placed at higher interest rate levels. Intuitively, these findings reflect the bidder's concern of ending with insufficient liquidity, i.e., the so-called loser's nightmare (cf. Simon, 1994). Here again, the predictions of our theoretical framework are largely supportive of the evidence. We identify, however, also a new effect that might shed additional light on incentives for bid shading in discriminatory auctions. Third, during relatively calm periods, bidders in ECB repo auctions have tended to submit excessively flat bid schedules at the level of the expected stop-out rate. We show here that the slope of bid schedules is "nearly" vanishing when bidders face little uncertainty and the liquidity of collateral assets is high. Again, this matches the evidence. Finally, we discuss another observation, which is the unexpected increase in the so-called Eonia spread with the introduction of an adapted implementation framework in early 2004. Also here, the predictions are in line with the evidence. Specifically, we show that a ceteris paribus increase in the size of the auction will lead to a wider spread between conditions in the primary and secondary money markets.

We go on to study the central bank's decision on the pricing rule. Empirically, the ECB seems to have a clear preference for using discriminatory pricing in its main refinancing operations. Indeed, the uniform-pricing rule has been employed for the main operations only in early 1999. To explore this issue, we determine the revenue impact of the pricing rule, and find that in the identified equilibria, expected revenue is strictly higher in the discriminatory auction than in uniform-price auction. This is a somewhat unexpected result because the stronger bid shading in the discriminatory auction had sometimes been understood to actually reduce the auctioneer's expected profits. We also show that the difference may be even more pronounced when required reserves are, so the established terminology, remunerated at the marginal rate, which is the case in the euro area. Finally, we mention another advantage of the discriminatory format, which is related to the signaling role of tender rates.

A remarkable piece of evidence was revealed when market turmoil triggered by the U.S. subprime crisis hit the euro money market. Specifically, it was found that following August 2007, counterparties of the Eurosystem would be willing to pay a premium above benchmark rates for participating in the auction. We apply our formal framework to comment also on these developments.

The rest of the paper is structured as follows. Section 2 reviews the empirical evidence. Section 3 outlines the auction model. In Sections 4 and 5, we characterize bidding equilibria of the uniform-price and discriminatory auctions, respectively, covering the cases of few and many bidders. Section 6 relates our predictions for many bidders to empirical observations for the euro area, while Section 7 discusses the Eurosystem's potential motivation for using the discriminatory format. Section 8 reviews some related theoretical literature. In Section 9, we derive predictions for bidding under market distress. Section 10 concludes. Appendices A through D provide, respectively, background information on the euro money market, anecdotal evidence on bid schedules in the Eurosystem auctions, a mathematical description of the allotment rule, and formal proofs of the propositions.

2. Empirical evidence

This section reviews some evidence concerning the puzzles mentioned in the Introduction, i.e., concerning the spread between primary and secondary market conditions, the loser's nightmare, the flatness of bid schedules during calm periods, and the increase in the Eonia spread. We will report here only on those findings that appeared to us as most relevant given the purpose of the present discussion. For a more comprehensive picture, the interested reader is referred to the original contributions.

2.1. Discount in the primary market

Consistent with the objectives of the present analysis, we will focus on the period where the Eurosystem employed the discriminatory variable-rate tender for its main refinancing operations, i.e., following June 2000. During this period, the effective average interest rate paid by counterparties in the repo auctions has been the weighted average tender rate. Interbank conditions are more difficult to measure, so that several proxies are used. We know of two papers that test for a positive spread.

Ayuso and Repullo (2003) analyze data for the period June 2000 through September 2001. During this period, the Eurosystem conducted 63 variable-rate tenders with a 2-week maturity. Two alternative measures of the interbank rate have been employed,

[4] There are also few other frictions. For eligible counterparties, participation in refinancing operations is absolutely free of charges and does not require any specific skills. Moreover, the Eurosystem accepts as collateral a broad range of assets including also very illiquid assets such as credit claims and asset-backed securities. A counterparty that is unable to forward even such collateral is unlikely to obtain any funding at reasonable conditions.

[5] Specifically, a capital charge applies to all unsecured interbank lending outstanding on reporting dates. Cf. Bindseil et al. (2003).

[6] A similar view is taken by Craig and Fecht (2007) who write that "…banks participating in the main refinancing operations only try to provide the liquidity they really need for themselves–particularly for fulfilling of their own minimum reserve requirements–instead of bidding to offer larger parts of the liquidity in the interbank market."

[7] The spread in conditions is also not explained by *differences in collateral standards* between primary and secondary markets. Specifically, the wider class of collateral accepted by the central bank compared to the private market is no reason whatsoever to explain the *lower* rates in central bank tenders compared to the interbank market. To the contrary, the difference in collateral standards between primary and secondary money markets just reinforces the puzzling evidence.

[8] In the euro area, several effects may cause uncertainty about the allotment for competitive bidders. First, aggregate liquidity demand may change between the publication of liquidity conditions and the actual allotment. Second, there may be counterparties with varying needs for liquidity that are constrained due to a lack of suitable collateral and credit rating, and may therefore bid at rates that win with probability one. A third possibility is the central bank's discretion about the allotment volume (on this last point, see also Section 8 and Appendix A).

[9] In either auction, bidders submit demand schedules, and a stop-out rate is determined by equating demand and supply. Then, with uniform pricing, the bidders pay the stop-out rate, while with discriminatory pricing, bidders pay their own bid rates.

namely 1-week Euribor and Eonia.[10] The results were as follows. The average spread between marginal tender rate and average tender rate (i.e., the weighted average rate) has been 1.7 basis points. The spread of Euribor (Eonia) above the average tender rate has been 4 basis points (3 basis points). Moreover, the spread between Euribor (yet not Eonia) and the average tender rate was significantly different from zero during the considered period.

Neyer and Wiemers (2004) consider a somewhat longer data basis that covers the period until December 2003. All variable-rate tenders during this period had a maturity of two weeks. Given that the 2-week Euribor has been available since 15 October 2001, the authors calculate spreads of the weighted average rate over both the 1-week and 2-week Euribor. The results are as follows. For the full (restricted) sample of 138 auctions (87 auctions), the average spread of the 1-week Euribor (2-week Euribor) over the weighted average rate in the Eurosystem auctions was 3.1 basis points (2.2 basis points). In both cases, the null hypothesis of a non-positive spread has been rejected at a confidence level of 1%. The two Euribor spreads were also significantly different from zero when daily data is employed, and independent of whether special events are accounted for.

We conclude that the available evidence conclusively suggests the existence of a discount in Eurosystem auctions between June 2000 and December 2003.[11]

2.2. The loser's nightmare

An intriguing issue is the response of bidders in Eurosystem auctions to an increase in uncertainty. We focus here on the impact of market volatility on the level of bids and bid dispersion in the main refinancing operations. Three recent contributions shed light on this issue.

Bindseil et al. (2005) evaluate bidding data for 53 main refinancing operations during the period June 2000 through July 2001. All operations in the sample had a maturity of two weeks and were conducted in the discriminatory format. The benchmark for secondary market conditions has been the 2-week Eonia swap rate. It is found that an increase in swap volatility by one basis point decreases the spread of the swap over the weighted average bid rate (the weighted average winning rate) by 0.374 basis points (0.526 basis points), and increases bid dispersion, measured as the quantity-weighted standard deviation of bid rates, by 0.043 basis points. These effects are reported to be significant. Thus, as the volatility of the swap rate increases, banks appear to bid more aggressively, i.e., higher relative to the benchmark. Bids are also somewhat more dispersed. The results do not change when regressions are re-run with a portion of the highest bids discarded from bid schedules. Also the introduction of inter-bidder dispersion as an additional explanatory variable does not invalidate the findings. The authors conclude that private information and the winner's curse are not driving bidding behavior in the Eurosystem's main refinancing operations.

Bruno et al. (2005) analyze panel data covering 59 main refinancing operations during the period July 2000 through August 2001. All operations had a maturity of two weeks, and were conducted under the discriminatory pricing rule. It is found that the volatility of the 2-week Eonia swap series has a positive and highly significant impact on the spread between average bid rate and minimum bid rate in the main refinancing operations. That is, as swap volatility increases, bidders are placing bids at higher interest rate levels. Moreover, volatility has also been found to have a significant positive effect on bid dispersion.

The paper by Linzert et al. (2007) is concerned primarily with longer-term refinancing operations. The data set covers 50 auctions conducted in the period March 1999 through May 2003 as pure variable-rate tenders, i.e., without a minimum bid rate. Pricing has been discriminatory in these operations. Uncertainty is proxied by the implied volatility derived from options on the 3-month Euribor. In contrast to the findings on the main refinancing operations, an increase in volatility turns out to decrease the weighted average bid rate significantly (and likewise bid dispersion). Thus, as volatility increases, bidders are placing their bids at lower interest rate levels. The authors conclude that the private information component of refinancing is more pronounced in longer-term refinancing than in the main refinancing operations.

In sum, the evidence suggests that private information about the common value of liquidity is not a central feature of the Eurosystem's main refinancing operations. Rather, uncertainty seems to drive up average bid levels in these tenders, both in absolute terms and relative to market rates. Also bid dispersion goes up in response to more uncertainty. Bidding in longer-term operations, however, seems to follow a different logic.

2.3. Flatness of bid schedules

A widely acknowledged phenomenon is that in calm markets, bid schedules in ECB repo auctions are often concentrated on very few grid points. Cassola et al. (2007) consider a data set covering 31 main refinancing operations during the period March 2004 through October 2004. The sample period was characterized by a stable minimum bid rate of 2%, and by an absence of almost any interest rate expectations, end-of-year effects, or other potential causes of market volatility. Indeed, the descriptive statistics of the data captures the characteristics of an extremely calm period. Stable secondary market conditions are reflected by an average Eonia swap rate of 2.0306% and an average market repo rate of 2.011%. Bidders had also relatively little uncertainty about the total allotment, which fluctuated somewhat around an average of EUR 239 bn. The number of bidders was 359 on average, which compares to an average of only 515 bids per auction. In line with the degenerated shape of bid schedules, the average marginal tender rate (weighted average rate) was at 2.007% (2.0148%). However, the relatively modest average bid-to-cover ratio of 1.26 suggests that even during this period, the remaining uncertainty induced bidders to compete in the interest rate dimension rather than by merely overbidding at the margin.

2.4. Eonia spread increased

In spring 2004, changes to the operational framework have been implemented that implied a significant increase in the volume of the Eurosystem's main refinancing operations. Initially, there was also a significant reduction in the uncertainty concerning market rates and supply surrounding the auctions because the ECB decided to provide more information and because expectations of rate changes were in principle no longer relevant for the bidding. From fall 2004 onwards, however, there was a clearly visible increase in the spread between Eonia and the minimum bid rate. Hassler and Nautz (2007) report that the median of the Eonia spread, i.e., the spread between Eonia and the minimum bid rate, has increased from 5 basis points before the changes to 8 basis points afterwards. In response to the widened spread, the ECB decided to slightly increase the allotment in the regular operations.

3. The model

To examine the evidence reviewed above, we will consider now a formal set-up with a central bank and finitely many counterparties. Each counterparty (or simply bank) will be endowed with a positive or negative balance on its reserve account. This balance may (but need

[10] The 2-week Euribor was not yet available at that stage.

[11] In fact, more evidence is available. Bindseil et al. (2005) have the interesting finding that an increase in the swap spread over the minimum bid rate by one basis point leads to an increase in the average bid by strictly less than one basis point. In a similar vein, Bruno et al. (2005) show that bidders shade their bids below market rates by about a quarter of the swap spread. Supportive evidence is also provided in ECB (2001).

not) be defined relative to exogenous reserve requirements. The bank minimizes net funding costs (or equivalently, maximizes net interest income) subject to the condition that it must clear the balance. Liquidity can be obtained either in the form of an allotment in the central bank auction or through interbank loans. We will assume that the auction precedes the trading, so any open position remaining after the auction must be closed using the interbank market.

As discussed in the Introduction, one key element of our theory is the explicit modeling of collateral requirements. Our specific assumptions in this dimension are motivated by the institutional environment of the euro money market. Two sorts of collateral will be available, a liquid one that can be used both in the primary and in the secondary market, and an illiquid one that can be used only in the primary market. We will assume that all credit vis-à-vis the central bank must be fully collateralized. In the private market, this will not be a requirement, but funding via secured lending may involve lower interest rates.

Against this back-drop, we develop the model. Altogether four perfectly divisible assets have a role in the model: central bank reserves (“liquidity”), liquid collateral, illiquid collateral, and net interest. For simplicity, we will assume that net interest payments are settled at a somewhat later stage so that these cash flows do not interfere with the management of liquidity in the current reserve maintenance period. There is a finite population of $N \geq 3$ risk-neutral banks. Each bank's initial liquidity position may be positive (a demand), zero, or negative (excess liquidity). For simplicity, we assume that the population of banks decomposes into two groups, banks $i = 1,\ldots, n$ with liquidity demand $q_i^0 > 0$ and banks $k = n+1,\ldots, N$ with zero or excess liquidity $\tilde{q}_k^0 \leq 0$, where $2 \leq n \leq N-1$. Thus, there are at least two counterparties with a liquidity deficit and there is at least one counterparty with excess liquidity.

The timeline is divided by five dates. At date 0, the central bank makes a public announcement of the competitive allocation of a neutralizing quantity of liquidity, where the total allotment

$$\tilde{Q} = \sum_{i=1}^{n} q_i^0 + \sum_{k=n+1}^{N} \tilde{q}_k^0 \tag{1}$$

will be known with certainty only at date 2. Bids are submitted at date 1. Commercial banks are informed about their allotments at date 2, and must then transfer an amount of collateral equal to the allotment to the central bank. At date 3, liquidity may be exchanged against collateral in the interbank market. Finally, at date 4, all secured transactions mature, i.e., collateral and liquidity are transferred back, both vis-à-vis the central bank and in the secondary market. Moreover, also at date 4, net interest is paid, both on central bank operations and interbank transactions.

Each counterparty with a liquidity demand, i.e., each bank $i = 1,\ldots, n$ is assumed to possess sufficiently large endowments of either type of collateral. However, the opportunity costs for the bank of using liquid vs. illiquid collateral differ. The use of illiquid collateral generates no opportunity cost, while the use of liquid collateral causes a positive opportunity cost (in terms of foregone net interest). For $q_i \geq 0$, we denote by $c_i^{opp}(q_i) \geq 0$ bank i's exogenous marginal opportunity cost of using a total of $q_i = q_i^P + q_i^S$ of liquid collateral for refinancing purposes in either the primary or in the secondary market. Under the assumptions made, it is clearly a dominant strategy for each bank to use liquid collateral only in the interbank market, to avoid the opportunity costs in transactions vis-à-vis the central bank.

The secondary market will be modeled *in reduced form*, reflecting anecdotal evidence on interbank conditions.[12] Specifically, any bank $i = 1,\ldots,n$ with a remaining liquidity demand of $q_i^S > 0$ obtains funding up to the required amount at the marginal borrowing rate $r^S + c_i^{\text{risk}}(q_i^S)$, where $c_i^{\text{risk}}(q_i^S) \geq 0$ denotes a mark-up (covering, for instance, lenders' risk controlling) on the exogenous risk-free rate $r^S \geq 0$. We will write $c_i(q_i^S) = c_i^{\text{risk}}(q_i^S) + c_i^{\text{opp}}(q_i^S)$ for bank i's effective marginal cost of funding above the risk-free rate. Conversely, any bank $k = n+1,\ldots,N$ with excess liquidity earns, net of the mark-up, only r^S. For convenience, we assume that collateral pledged by a borrower of reserves does not generate any value for the cash-rich lender (in addition, of course, to securing the respective credit transaction).

It turns out to be useful at this point to derive bidders' marginal valuations for quantities obtained in the primary market. The intuition is as follows. As an unsuccessful bidder must make use of increasingly more expensive collateral in the secondary market, while inexpensive collateral can be used for funding obtained through the auction, marginal valuations will be strictly declining over a range of quantities. To keep the analysis simple, we will assume henceforth that $c_i^{\text{risk}}(.)$ and $c_i^{\text{opp}}(.)$ are linear with $c_i^{\text{risk}}(0) = c_i^{\text{opp}}(0) = 0$. In this case, $\partial c_i / \partial q_i$ is a constant, which we assume to be positive.

Proposition 1. *Fix some bank i with liquidity demand $q_i^0 > 0$. Then, with ex-post choice of collateral, bank i's marginal valuation for quantities $q_i^P \geq 0$ in the primary market is given by*

$$v_i\left(q_i^{\text{P}}\right) = \begin{cases} \bar{v}_i - \dfrac{q_i^{\text{P}}}{B_i} & \text{for } 0 \leq q_i^{\text{P}} \leq q_i^0 \\ r^S & \text{for } q_i^{\text{P}} > q_i^0, \end{cases} \tag{2}$$

where $\bar{v}_i = r^S + c_i(q_i^0)$ and $B_i = (\partial c_i / \partial q_i)^{-1}$.

Fig. 1 shows, among other things, the graph of a marginal valuation function for a counterparty with liquidity demand. The interest rate $\bar{v}_i$ corresponds to the maximum rate that counterparty i, when relying exclusively on secondary market funding, would be willing to pay for the first marginal unit of credit from the central bank, collateralized by illiquid collateral. A higher liquidity demand q_i^0 shifts bank i's marginal valuation function rightward as more of the expensive liquid collateral needs to be used following an unchanged allotment. Similarly, a more abundant availability of liquid collateral (i.e., a higher B_i) makes marginal valuations respond in a less pronounced fashion to the allotment in the tender, so that marginal valuations will be flatter.

For a counterparty k with excess liquidity $\tilde{q}_k^0 \leq 0$, the market environment implies constant marginal valuations. This is quite useful because it allows us to abstract from potential entry by liquidity-rich banks. Indeed, unless the stop-out rate in the auction falls below r^S, banks with excess liquidity cannot earn any profits from participating in the auction. We may therefore assume without much loss of generality that only banks $i = 1,\ldots,n$ choose to participate. Moreover, for simplicity, we assume that n is common knowledge among market participants.[13]

Next, we describe the auction itself. The tender mechanism asks each counterparty i to submit a schedule of cumulated bids that specifies, for any interest rate $r \geq r_{\min}$, a quantity $x_i(r) \geq 0$ that bidder i is willing to buy at rate r. Here, $r_{\min}$ is the central bank's reserve price (or minimum bid rate). For expositional simplicity, we will assume throughout that $r^S = r_{\min}$.[14] A schedule $x_i(.)$ is called *admissible* if $x_i(.)$ is non-increasing, left-continuous, and if $x_i(r) = 0$ for any sufficiently high r.[15] Only admissible bid schedules are accepted by the central bank. Let $x(r) = \sum_{i=1}^{n} x_i(r)$ denote the aggregate cumulated bid at interest rate r.

We assume that the liquidity positions $\tilde{q}_k^0$ of banks $k = n+1,\ldots,N$ with excess liquidity are unobservable for the bidders at date 1.

[12] We conjecture that a model of reserve management in the tradition of Poole (1968) would lead to similar conclusions.

[13] Empirically, there is ample evidence that a bidder with a liquidity demand (with excess liquidity) is more (less) likely to participate in the Eurosystem's main refinancing operations (see Bruno et al., 2005, Craig and Fecht, 2007, and Fecht et al., 2007).

[14] This assumption can be dropped when the total allotment is suitably bounded relative to bidders' aggregate demand.

[15] By definition, the bid schedule $x_i(.)$ is left-continuous if $\lim_{r' \to r,\, r' < r} x_i(r') = x_i(r)$ for all r.

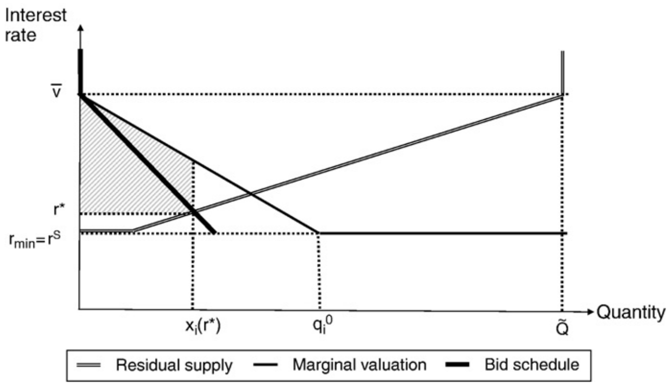

Fig. 1. Marginal valuations and equilibrium in the uniform-price auction.

Instead, bidders hold a common belief about the statistical distribution of the total allotment $\tilde{Q}$ following date 0. For a given realization of $\tilde{Q}$, let $R^*(\tilde{Q})=\{r\geq r_{\min}|x(r)\leq\tilde{Q}\}$ denote the set of interest rates at which aggregate cumulated bids can be satisfied with $\tilde{Q}$. It is straightforward to check that $R^*(\tilde{Q})$ is non-empty for any $\tilde{Q}\geq 0$ and for any vector of admissible bid schedules $(x_1(.),\ldots,x_n(.))$. We may therefore define the *stop-out rate* as the infimum $r^*(\tilde{Q})=\inf R^*(\tilde{Q})$ of such interest rates. The allotment $q_i^*(\tilde{Q})$ to bidder $i=1,\ldots,n$ is then determined by satisfying all bids strictly above the stop-out rate, and by applying rationing at the margin, if necessary.[16] For $\tilde{Q}$ given, the resulting tuple $(r^*,q_1^*,\ldots,q_n^*)$ consisting of stop-out rate and individual allotments will be referred to as the *outcome* of the tender.

4. Uniform pricing

This section derives the equilibrium in the auction with uniform pricing. The uniform-price auction is very well accessible to the mathematical analysis, and will mainly serve as a point of comparison.

In the uniform-price tender, bidder i pays the stop-out rate r^* per marginal unit, so that bidder i's net interest from an outcome $(r^*,q_1^*,\ldots,q_n^*)$ is given by

$$\Pi_i^u = \int_0^{q_i^*}\{v_i(q_i)-r^*\}dq_i.$$

Fig. 1 illustrates bidder i's net interest under the uniform pricing rule as the shaded area between marginal valuation and stop-out rate. The optimal schedule for bidder i is determined by the uncertainty about the residual supply curve. In our model, supply is perfectly inelastic, while bid schedules are downward sloping. Hence, the residual supply for a given bidder, defined as the horizontal difference between supply $\tilde{Q}$ and the aggregate of the bid schedules submitted by other counterparties, must be increasing in the interest rate.

To ensure tractability, we will assume henceforth a common maximum valuation $\bar{v}=\bar{v}_1=\ldots=\bar{v}_n$. Fig. 1 suggests that it will then be suboptimal for a counterparty to bid a strictly positive quantity at $r>\bar{v}$. On the other hand, it is clear that a zero quantity bid at any interest rate $r<\bar{v}$ will be dominated. Therefore, given the linear set-up, a natural candidate for an equilibrium strategy is to scale down actual demand $d_i(r)=B_i\max\{\bar{v}-r,0\}$ by a constant factor, so that the bid schedule attains the form

$$x_i(r) = B_i^u max\{\bar{v}-r,0\} \quad (3)$$

for some constant $B_i^u>0$. In the context of a uniform-price auction, an equilibrium in which all bidders $i=1,\ldots,n$ use some schedule of the form (3) will be referred to as a linear equilibrium. Motivated by the anecdotal evidence on bid curves offered in Appendix B, but also by restrictions imposed by tractability, we will focus the analysis on the linear case.

To derive the equilibrium, we follow the approach used by Kyle (1989), Klemperer and Meyer (1989), Back and Zender (1993), and others, which relies on the intuition that if the quantity to be transacted is uncertain for the bidders, then the optimal bidding strategy for the uniform pricing rule can essentially be found by a state-by-state optimization against the ex-post residual supply curve. We will assume in the sequel that $\tilde{Q}$ has full support on some interval $[0;\bar{Q}]$, where $\bar{Q}>0$ is not too large. The support restriction is useful because it ensures that the stop-out rate does not fall to the minimum bid rate in equilibrium with strictly positive probability, which would engender overbidding at the minimum bid rate. We allow, however, for the possibility that the stop-out rate drops to the minimum bid rate as the result of a deviation by an individual bidder.

Proposition 2. *Assume $n\geq 3$, and that $\bar{Q}$ is not too large.*[17] *Then there exists a linear equilibrium in the auction with uniform pricing. In fact, the equilibrium is unique within the class of linear equilibria. When compared to actual demand, bids are shaded, i.e., $B_i^u<B_i$ for all i. Moreover, in any equilibrium with heterogeneous bidders, shading of bids is isotone, i.e., for all $i\neq j$ we have $B_i^u<B_j^u$ if and only if $B_i<B_j$. In the symmetric set-up,*[18] *the equilibrium is given by $B_i^u/B_i=(n-2)/(n-1)$ for all i.*

The prediction of the model is consistent with general studies of bidding behavior in uniform-price auctions such as Ausubel and Cramton (2002). Specifically, the bidder has an incentive to shade actual demand because the stop-out rate will apply not only to the marginal unit, but to the entire allotment to bidder i. In fact, bid shading will be differential, i.e., shading will be more pronounced at larger quantities than at smaller quantities. This is only natural because for large quantities, the overall price impact will be much stronger than for small quantities.

Next, we consider the case of a large auction by letting the number n of bidders go to infinity. Throughout the analysis, it is tacitly understood that the total number of counterparties $N=N(n)$ depends on n and satisfies $N(n)\geq n+1$. For the uniform auction, the following result is obtained.

Proposition 3. *Consider a family of auctions $\{T(n)\}_{n=3,4,5\ldots}$ with uniform pricing, such that in auction $T(n)$, a random quantity not larger than $\bar{Q}(n)=n\bar{q}$ is auctioned off to n bidders, where $\bar{q}>0$. Assume that the slope parameters $\{B_i(n)\}_{i=1,\ldots,n}$ are uniformly bounded, i.e., there are $\bar{B}>\underline{B}>0$ such that $\underline{B}\leq B_i(n)\leq\bar{B}$ for all $n\geq 3$ and for all $i=1,\ldots,n$. Assume also that $\bar{q}<\underline{B}(\bar{v}-r_{\min})$. Then $\lim_{n\to\infty}B_i^u(n)/B_i(n)=1$ for all i.*

Thus, provided that relative marginal valuations do not vary too widely across bidders, bid shading is predicted to disappear under the uniform pricing rule in large populations of bidders. This should be intuitive because with an increasing number of bidders, the residual supply curve faced by an individual bidder becomes flatter and flatter in the (q, r) diagram. As a consequence, the effect of the individual bid schedule on the stop-out rate will be smaller and smaller, inducing the bidder to ask at a given interest rate for a quantity that is larger and closer to true demand. In the limit, the residual supply curve is essentially a horizontal line, so that individual bidders find it in their own interest to behave like price-takers, i.e., to submit their true demand.

[16] A mathematical description of the allotment rule can be found in Appendix C.

[17] For just two bidders, there is no linear equilibrium with strictly decreasing bid schedules. In this case, the residual supply curve is too steep to allow convergence of the dynamics of mutual best responses.

[18] By a symmetric set-up, we mean a parameter constellation satisfying $B=B_1=\ldots=B_n$. This implies, in particular, a common liquidity deficit $q^0=q_1^0=\ldots=q_n^0$.

5. Discriminatory pricing

This section discusses bidding behavior in the auction with the discriminatory pricing rule.

The reader will have noted that bid schedules can be formally described in two ways, one expressing demand at given interest rates (the one used so far), and the other attaching interest rates to given quantities. The discriminatory pricing rule requires the payment of the individual bidder's own interest rate bid on the allotted quantities. It is therefore natural to work, rather than with the bid schedule $x_i(.)$ itself, with the inverse schedule $b_i(.)$ defined via $b_i(q_i) = \inf\{r \geq r_{\min} | x_i(r) \leq q_i\}$. The figure $b_i(q_i)$ can then be understood as the stated willingness to pay (the "bid") for the marginal unit at quantity q_i, contrasting the true willingness to pay for the marginal unit (the "valuation"), which is given by $v_i(q_i)$. Similarly as in the definition of the stop-out rate, one can check that $b_i(q_i)$ is well-defined for any admissible bid schedule $x_i(.)$ and any $q_i \geq 0$. Moreover, $b_i(.)$ is non-increasing and right-continuous.

Under discriminatory pricing, bidder i pays his own bid $b_i(q_i)$ for any marginal unit, so that the resulting net interest from outcome $(r^*, q_1^*, \ldots, q_n^*)$ amounts to

$$\Pi_i^d = \int_0^{q_i^*} \{v_i(q_i) - b_i(q_i)\} dq_i. \tag{4}$$

The integral corresponds to the shaded area in Fig. 2 that illustrates the equilibrium in the discriminatory auction. Shown is one realization of the residual supply curve. The intersection point between counterparty i's downward-sloping bid schedule and the residual supply curve determines simultaneously the stop-out rate r^* and the allotment $q_i^* = x_i(r^*)$ to bidder i. In contrast to the uniform-price auction, the entire fulfilled part of the bid schedule determines the bidder's profit, not just the quantity demanded at the stop-out rate. This feature of the discriminatory auction makes the general characterization of the equilibrium more involved so that we have to restrict ourselves to the symmetric set-up.

Moreover, to obtain a linear equilibrium also under the discriminatory pricing rule, a functional assumption on the uncertainty will be necessary. Fortunately, the relevant class of distributions (those with a linear hazard ratio) captures relatively well a bidder's residual uncertainty about supply. Specifically, a random variable $\tilde{Q}$ is considered that is distributed on the interval $[0; \overline{Q}]$ according to the density

$$f(\tilde{Q}) = \frac{\alpha}{(\overline{Q} - \tilde{Q})^{1-\alpha} \overline{Q}^{\alpha}},$$

where $\alpha > 0$. For $\alpha = 1$, the distribution is uniform. For $\alpha < 1$, the distribution of $\tilde{Q}$ is skewed towards its "mode" $\overline{Q}$, which should better correspond to the institutional situation in the euro area.

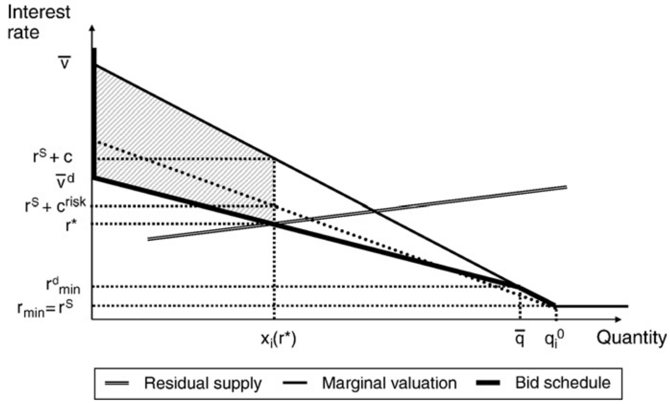

Fig. 2. Equilibrium in the discriminatory auction.

The survey in Section 2 has shown that bidding behavior in the euro area depends to a certain extent on the degree of uncertainty for the bidders. As mentioned in the Introduction, we will be mainly interested in the case where there are many bidders and uncertainty is non-negligible. In this case, it is plausible to assume that the interest rate grid does not interfere too much with the bidder's problem, and may therefore be neglected. Equilibrium bidding is then characterized as follows.

Proposition 4. *Assume $n \geq 2$, and that bidders $i = 1, \ldots, n$ have identical marginal valuations $v_i(q_i) = \max\{\overline{v} - q_i/B, r^S\}$. Then, provided that $\alpha > 1/(n-1)$, there exists an equilibrium in the auction with discriminatory pricing in which bidder $i = 1, \ldots, n$ submits the piecewise linear bid schedule*

$$x_i(r) = \begin{cases} 0 & \text{for } r > \overline{v}^d, \\ B^d(\overline{v}^d - r) & \text{for } r^d_{\min} < r \leq \overline{v}^d, \\ B(\overline{v} - r) & \text{for } r_{\min} \leq r \leq r^d_{\min}, \end{cases} \tag{5}$$

where

$$\overline{v}^d = \overline{v} - \frac{\overline{Q}}{(n + \alpha(n-1))B}, \tag{6}$$

$$B^d = \left(1 + \frac{n}{\alpha(n-1)}\right)B, \tag{7}$$

$$r^d_{\min} = \overline{v} - \frac{\overline{Q}}{nB}, \tag{8}$$

are the maximum interest rate bid, the absolute slope of the inverse bid schedule, and the minimum stop-out rate, respectively.

Figs. 1 and 2 jointly illustrate how and why bid shading differs across auction formats. With uniform pricing, bid schedules are steeper than actual demand, because the level of inframarginal bids is without consequence for the bidder. With discriminatory pricing, however, bid schedules are flatter than actual demand, because there is little competition for inframarginal quantities. In Section 7, we will explore the implications of this point for the central bank's preferences concerning the pricing rule. The shape of bid schedules under discriminatory pricing, taken for itself, will play an important role in the next section when we compare the predictions of the model with the evidence.

Equilibrium bidding differs between the two auction formats also when there are many bidders. Specifically, assume the quantity allotted is $\overline{Q}(n) = n\overline{q}$. Then, for an increasing number of bidders the maximum interest rate at which a positive bid is placed in the discriminatory auction converges from below against

$$\lim_{n \to \infty} \overline{v}^d(n) = \overline{v} - \frac{\overline{q}}{(1 + \alpha)B}. \tag{9}$$

Thus, competition does move up the level of bids, but the effect is not strong enough to eliminate bid shading in the limit. This point, as well, will be taken up in the next section.

6. The case of the Eurosystem

This section compares the predictions for the discriminatory auction with the empirical evidence. The topics surveyed in Section 2 will be treated one by one, keeping the original order. To ensure the validity of the equilibrium characterization illustrated in Fig. 2, it will be assumed henceforth that $\alpha > 1/(n - 1)$.

6.1. Primary vs. secondary market funding

Let $\tilde{q} = \tilde{Q}/n$ denote the per-bidder allotment in the auction with n bidders. In the absence of informational frictions, interbank conditions following the allotment $\tilde{q}$ should roughly reflect counterparties' marginal willingness to pay for credit. We will therefore use the marginal valuation $v(\tilde{q}) = r^S + c(q^0 - \tilde{q})$ as a proxy for (deposit or swap) interbank conditions, where we dropped the reference to the individual bidder i for notational convenience.[19] For conditions in the primary market, the following terminology is adapted from Bindseil et al. (2005). Denote by $r_d^{\text{mar}}(\tilde{q}, n)$ the marginal interest rate (i.e., the stop-out rate) in the discriminatory auction. Then, *bid shading* $v(\tilde{q}) - r_d^{\text{mar}}(\tilde{q}, n)$ is defined as the difference between marginal valuation and marginal interest rate. Similarly, let *underpricing* $v(\tilde{q}) - r_d^{\text{war}}(\tilde{q}, n)$ be the difference between the marginal valuation and the quantity-weighted average winning rate $r_d^{\text{war}}(\tilde{q}, n) = \tilde{q}^{-1} \int_0^{\tilde{q}} b(q)dq$ in the discriminatory auction, where $b(.)$ is the inverse bid schedule used in equilibrium by all bidders. Finally, the *discount* $v(\tilde{q}) - \overline{q}^{-1} \int_0^{\overline{q}} b(q)dq$ is the difference between marginal valuation and the average equilibrium bid rate, where we ignore the part of the bid schedule that is allotted with probability zero.

Proposition 5. *In the linear equilibrium of the discriminatory auction with n bidders, as described in Proposition 4, expected bid shading, expected underpricing, and the expected discount are all strictly positive. Moreover, these spreads do not vanish in the limit as $n \to \infty$.*

To understand why bid shading does not disappear in the limit in the discriminatory auction, it may be instructive to consider the bidder's marginal costs of increasing the expected allotment above the equilibrium amount. In equilibrium, these marginal costs must be the same for the (symmetric) uniform-price and discriminatory auctions. Now, intuitively, to increase the expected allotment, the bidder would have to raise the whole bid schedule. In fact, given that residual supply is flatter in the discriminatory auction, we would expect that the rise of the bid schedule is more pronounced in the case of uniform pricing.[20] Nevertheless, and this is the crucial point to note, *the bidder's expected costs of raising the bid schedule by, say one basis point, is much lower under the uniform pricing rule than under the discriminatory pricing rule.* Indeed, the expected stop-out rate under the uniform-pricing rule would increase by much less than one basis point, while the price impact is immediate in the case of the discriminatory auction. Moreover, this effect becomes relatively stronger as the number of bidders increases.

6.2. The loser's nightmare

The evidence reviewed in Section 2 suggests that bidders in Eurosystem auctions have an incentive to increase both the level of their bids and bid dispersion when uncertainty increases. In the model, it is the parameter α that measures the uncertainty about the allotment and equivalently about the marginal valuation. Indeed, the normalized standard deviation of the allotment

$$s = \frac{\sqrt{E\left(\tilde{q}^2\right) - E(\tilde{q})^2}}{E(\tilde{q})} = \sqrt{\frac{\alpha}{2+\alpha}}$$

is strictly increasing in α. Similarly, the variance of the marginal valuation

$$\sigma^2 = E\left(v(\tilde{q})^2\right) - E(v(\tilde{q}))^2 = \frac{\alpha\overline{q}^2}{(1+\alpha)^2(2+\alpha)B^2}$$

is increasing in α provided that $s<0.48$.[21] The following comparative statics result can be obtained.

Proposition 6. *Consider a ceteris paribus increase in $\alpha<1$. Then the average bid level (ignoring bids that are allotted with probability zero), the expected stop-out rate, and the expected average winning rate all go up. However, the expected marginal valuation increases even more strongly, so that expected bid shading, expected underpricing, and the expected discount all go up as well. The strict monotonicity is valid throughout also in the limit as $n \to \infty$.*

With an increase in uncertainty α, the middle section of the equilibrium bid schedule, i.e., the section corresponding to positive allotments up to $\overline{q}$, moves clockwise around its right end point. This is intuitive because bids placed at lower quantities are more likely to be competitive. In addition to higher (and more dispersed) bidding, however, it becomes more likely that the allotment will be smaller than expected, and that the realized marginal valuation is larger. This widens the spread between marginal willingness to pay for credit and actual bids. Thus, concerning the impact of volatility, the theoretical framework explains an increasing absolute level and dispersion of bids, but is in conflict with bid levels increasing relative to secondary market conditions.[22]

6.3. The shape of bid schedules

Quite generally, the discriminatory pricing rule implies that bid schedules are flatter than marginal valuations. To investigate the determinants of the slope of the bid schedule, define *bid dispersion* Δ as the quantity-weighted standard deviation of the bid schedule, where we ignore bids that are allotted with probability zero (as in the definition of the discount). Formally, this definition amounts to

$$\Delta^2 = \int_0^{\overline{q}} b(q)^2 dq - \left(\int_0^{\overline{q}} b(q)dq\right)^2.$$

A simple calculation shows that the bid dispersion in the identified equilibrium of the discriminatory auction is given by

$$\Delta_d = \sqrt{\frac{\overline{q}}{12B}\frac{\alpha(n-1)}{n+\alpha(n-1)}}. \tag{10}$$

This formula implies that, for a given number of bidders, if the market environment becomes more and more predictable, i.e., if α becomes smaller, bid schedules become less and less dispersed. Furthermore, bid dispersion is declining in the liquidity of collateral assets. All this is intuitive and consistent with the evidence.

The present discussion also allows seeing more clearly why some uncertainty is needed to sustain the equilibrium in Proposition 4. Intuitively, the problem is that if uncertainty about the allotment diminishes ceteris paribus, bidders would submit excessively flat bid schedules. However, at some stage, provided that no bidder submits bids above her marginal valuation, there would be an incentive for a single bidder to undercut the standing bids of all the other bidders, which would unravel the equilibrium. When individual bidders are

[19] The allotment also determines the secured market rate $r^S + c^{\text{risk}}(q^0 - \tilde{q})$, which may be higher or lower than r^*, depending on the parameters of the model.

[20] This point can be verified without much additional effort in the special case $\alpha = 1$.

[21] This restriction should be satisfied in the euro area where allotments have been relatively well-predictable most of the time.

[22] A potential explanation might be credit rationing that would drive a wedge between marginal valuations and secondary market rates (see also Section 9).

small and uncertainty about the allotment is non-negligible, however, the equilibrium is sustainable.

6.4. The increase in the Eonia spread

The evidence surveyed in Section 2 suggests that an increase of the tender size might imply an increased risk of ending with insufficient liquidity. The problem for commercial bank liquidity management is then that with a higher probability, liquid collateral may become relatively scarce. Intuitively, this should cause an increase of interest rates in the secondary market. The following result can be obtained.

Proposition 7. *Consider a simultaneous and proportional inflation of* $\bar{q}$ *and* q^0. *Then, in the considered equilibrium of the discriminatory auction, the expected stop-out rate, the expected average winning rate, the expected marginal valuation, and the standard deviation of the marginal valuation all increase. Moreover, expected bid shading, expected underpricing, the expected discount, and bid dispersion all increase. The respective strict monotonicity holds also in the limit as* $n \to \infty$.

In particular, Proposition 7 predicts that a rise in the tender size driven by exogenous changes in the operational framework should increase expected rates both in the tender and in the interbank market. Such an exogenous change might have occurred when the Eurosystem modified its operational framework in spring 2004, with an accompanying effect of approximately doubling the size of its weekly main refinancing operations.

7. Discriminatory vs. uniform pricing

What are the relative merits of using uniform vs. discriminatory pricing in central bank tenders? This section will compare the two auction formats with respect to expected revenues, expected costs of remuneration (as explained below), and signaling.

In the dimension of expected revenues, our earlier findings support the intuition that for many bidders, declining valuations, and aggregate uncertainty, the uniform-price auction resembles a second-price auction for an indivisible good, which should induce bidders to submit their true valuations. Yet regarding the revenue comparison, there is a countervailing effect that for relatively large allotments, discriminatory pricing generates higher revenues than uniform pricing. As our next result will show, this latter effect always dominates the bid shading in our framework. To state the proposition, we introduce the following notation. For a given per-bidder allotment $\tilde{q}$, let $\pi^u(\tilde{q},n) = n\tilde{q}r_u^{\text{war}}(\tilde{q},n)$ and $\pi^d(\tilde{q},n) = n\tilde{q}r_d^{\text{war}}(\tilde{q},n)$, respectively, denote the realized revenues in the uniform-price and discriminatory auctions, where $r_u^{\text{war}}(\tilde{q},n)$ is defined for the uniform-price auction in analogy to $r_d^{\text{war}}(\tilde{q},n)$.

Proposition 8. *Consider the symmetric linear equilibria described in Propositions 2 and 4. Then for* $n \geq 3$, *we have* $E(\pi^d(\tilde{q},n)) > E(\pi^u(\tilde{q},n))$. *The inequality is strict also in the limit as* $n \to \infty$.

Thus, in our model, the discriminatory auctions yields strictly higher expected revenues than the uniform-price auction.

It must be noted, though, that in the case of the Eurosystem auctions, central bank income differs from revenues. Background is the institutional peculiarity that in the euro area, in contrast to the reserve system used so far in the U.S., required reserves are *remunerated*. The idea here is to reimburse commercial banks the cost of holding liquid means equivalent in size to minimum reserves. In the euro area, remuneration is based on the marginal tender rate. To study the impact of remuneration formally, let $L \geq 0$ denote counterparties' common level of minimum reserves. Remuneration implies that following a total allotment of $\tilde{Q}$, each counterparty obtains an additional net interest of $Lr^*(\tilde{Q})$. We are interested therefore in the relative level of the expected stop-out rate under the two auction formats.

Two effects determine equilibrium remuneration. First, there is a direct impact of the pricing rule on the average level of the marginal tender rate. It turns out that also with remuneration, the marginal tender rate is typically higher with uniform pricing, unless the market environment is relatively calm. Second, the remuneration itself increases the average level of bids. In our model, this effect is always stronger with uniform pricing than with discriminatory pricing. Clearly, the second, strategic effect disappears as the number of bidders grows larger. The following result can be obtained:

Proposition 9. *Consider a symmetric set-up with* $n \geq 3$ *bidders. Then, with remuneration of individual reserve requirements* $L \geq 0$ *at the level of the stop-out rate, there exist linear equilibria in the uniform-price and discriminatory auctions generalizing those described in Propositions 2 and 4. Specifically, with remuneration, bidders bid as if* $\bar{v}$ *were replaced by* $\bar{v}(L)$, *where*

$$\bar{v}(L) = \begin{cases} \bar{v} + \dfrac{L}{(n-2)B} & \text{for uniform pricing;} \\ \bar{v} + \dfrac{\alpha L}{B(n+\alpha(n-1))} & \text{for discriminatory pricing:} \end{cases}$$

In these equilibria, for $n \geq 4$, *there exists a threshold value* $\alpha^* \in (0;1)$ *such that* $E(r_u^{mar}(\tilde{q},n)) > E(r_d^{mar}(\tilde{q},n))$ *if and only if* $\alpha > \alpha^*$. *Moreover, when* α *is fixed,* $E(r_u^{mar}(\tilde{q},n)) - E(r_d^{mar}(\tilde{q},n))$ *becomes strictly positive for sufficiently large n and does not vanish in the limit as* $n \to \infty$.

Thus, with aggregate uncertainty and many bidders, the expected remuneration payment might indeed be smaller for the central bank when the discriminatory format is employed.

While Propositions 8 and 9 suggest a revenue motive, there are alternative reasons for the monetary authority to rely on the discriminatory format. The most obvious one is that the large discriminatory auction produces stop-out rates that, compared to the uniform-price auction, are on average closer to the main policy rate r^S. Furthermore, the stop-out in the discriminatory format is less volatile than in the uniform-price format. These properties of the discriminatory auction may be relevant for the central bank when the market interprets the marginal tender rate as a signal concerning monetary policy. Our analysis might therefore explain why, even when revenue maximization is not the primary objective, the discriminatory pricing rule has been used almost exclusively in the main refinancing operations of the Eurosystem.[23]

8. Related theoretical literature

In his path-breaking paper on *auctions of shares*, Wilson (1979) offers an explicit analysis of the uniform-price auction under constant marginal valuations and incomplete information. Back and Zender (1993) characterize bidding equilibria also for the discriminatory auction, assuming likewise constant marginal valuations. This allows them to compare revenues between pricing rules and to comment on the policy discussion concerning U.S. Treasury auctions.[24]

Klemperer and Meyer (1989) consider a reverse auction in which profit-maximizing oligopolists select supply functions in the presence of uncertainty about market demand. Our discussion of auctions with uniform pricing in Section 4 is closely related to their analysis. Specifically, Proposition 3 extends a special case of Klemperer and

[23] Maybe interestingly, our findings are also consistent with the view that overly high prices produced by the Dutch pricing rule have been the main motivation for the Bundesbank to switch to American pricing in fall 1988 (cf. Nautz, 1997).

[24] For an overview over this debate, see also Bikhchandani and Huang (1993) and more recently, Goldreich (2007). Much of the research on multi-unit auctions relates to Friedman's (1960) famous advocacy of the uniform-price auction.

Meyer's Proposition 8a (linear demand with $m = 0$) by allowing for bidder heterogeneity. We also offer an explicit treatment of the non-negativity requirement for quantities. Moreover, our model allows for an optional (lower) price cap. Finally, Section 5 in our paper can be understood as an adaptation of Klemperer and Meyer's model to the case of discriminatory pricing. Ausubel and Cramton (2002) also assume declining marginal valuations and show in particular that the incentives for differential bid shading may cause an allocational inefficiency in the uniform-price auction. Our findings illustrate their analysis by discussing specific equilibria with decreasing returns in both the uniform-price and the discriminatory auction.[25]

The structure of the equilibrium prediction is known to change when the auctioneer adjusts supply in response to incoming bids (see Lengwiler, 1999; Back and Zender, 2001; McAdams, 2007). This practice, however, is apparently not applied by the Eurosystem (cf. Appendix A).

Independent characterizations of bidding behavior in the primary market that, while different in interpretation, are structurally similar to our results have been given by Röell (1998) and Biais et al. (2000) in the context of adverse selection and by Viswanathan and Wang (2002) in the context of risk aversion. Our eventual generalization to distributions with a linear hazard ratio in Section 5 draws on Viswanathan and Wang's work. Hortaçsu (2002) considers a share auction with discriminatory pricing for two bidders and an exponential distribution of types. For a heterogeneous population of bidders, Menezes and Monteiro (1995) show the impossibility of an equilibrium in a discriminatory bidding game.[26]

9. Predictions for the credit crunch

The euro money market was hit by the U.S. subprime crisis in the first half of August 2007. To investigate incentives for bidding during a liquidity crisis, we consider now a sudden exogenous development through which collateral assets held by commercial banks become less liquid, or equivalently, a development through which the use of liquid collateral causes higher marginal opportunity costs.

In the model, this type of exogenous shock corresponds to a decrease of collateral quality B. The ceteris paribus impact of this change would be, as suggested by Proposition 1, that banks with a net liquidity demand would have strongly increased marginal valuations for central bank credit. Moreover, marginal valuations of these banks would decline more sharply than before, reflecting the counterparty's primary goal of satisfying its minimum reserve requirement. The resulting predictions are as follows.

Proposition 10. *A ceteris paribus drop in collateral quality from B down to $B' < B$ has the same consequences on tender rates and marginal valuations as a simultaneous inflation of $\bar{q}$ and q^0 by the factor $B/B' > 1$.*

Thus, as in Proposition 7, the average level of tender rates goes up, there is a rise in marginal valuations, bids are more dispersed, and there is more volatility in marginal valuations. All this is largely in line with anecdotal evidence that is available at the time of writing, and suggests that the general conclusions that can be drawn from the model hold also during a turmoil period.

A puzzle remains, however. Specifically, towards the end of the year 2007, banks apparently were willing to pay a premium for an allotment in the ECB's repo auctions. More precisely, while tender rates went higher and higher, market benchmarks such as Eonia and 1-week Eonia swap remained at relatively moderate levels (cf. Cassola and Morana, 2008). Thus, during this period, bidders' marginal willingness to pay for credit following the tenders must have been much higher than suggested by secondary market rates.[27]

10. Conclusion

Refinancing operations are large-scale financial auctions that are used by central banks such as the ECB to provide an appropriate amount of reserves to the banking sector. The present paper has offered a model of refinancing operations in the context of a reduced-form market for interbank credit. In the model, the central bank invites a finite number of counterparties to submit schedules of bids, and only the highest bids will be successful. Counterparties that end up with insufficient funding from the auction must turn to the secondary market where credit can be obtained only against the most expensive sorts of collateral.

We have used the model to investigate the weekly main refinancing operations conducted by the Eurosystem following June 2000. Four stylized facts have been analyzed. First, concerning the spread between primary and secondary market conditions, we found that bid shading is persistent in large discriminatory auctions, provided that aggregate uncertainty about the allotment does not vanish in the limit. This suggests a new explanation of the interest rate spread. We also analyzed in some depth the intricate issue of how uncertainty influences bidding behavior in repo auctions. Here our model predicts that both the absolute level of bids and their dispersion increase in the volatility of the expected market rate. Also this finding is largely consistent with the evidence, i.e., with the hypothesis of the loser's nightmare. A third regularity that turned out to be reflected in the equilibrium prediction is the flatness of bid schedules during calm periods. Finally, we looked at the "mysterious" increase of the Eonia spread following March 2004, and can offer also here a theoretical prediction that is consistent with the evidence.

We then discussed the question why the ECB has been relying mostly on the discriminatory format in its main refinancing operations. We could show that with sufficiently many bidders, expected revenues are strictly higher under the discriminatory format than under the uniform-price format, despite the more pronounced bid shading. This revenue dominance may even become stronger when the central bank, as in the case of the Eurosystem, has chosen to reimburse interest paid on required holdings of reserves. However, we have also shown that tender rates in the discriminatory format are on average closer to the main policy rate (and less noisy), which might even be more relevant from a policy perspective.

Finally, motivated by the credit crunch following August 2007, we derived the implications of a deterioration of collateral quality on equilibrium bid schedules and the bidder's marginal willingness to pay for credit after the auction. Here we found the prediction that illiquidity of collateral not only increases the average level of both tender rates and marginal valuations, but also bid dispersion and the volatility of marginal valuations. These general conclusions suggest that the model can also be used to study bidding behavior during a liquidity crisis.

Appendix A. Institutional background

This appendix provides some basic information on the euro money market and the repo auctions conducted by the Eurosystem. For a more thorough description of the implementation framework in the euro area, the reader is referred to ECB (2006).

From the perspective of the Eurosystem, there is a well-defined set of market participants in the euro money market, the so-called *eligible*

[25] There exists a closely related literature on auctions of finitely many identical goods. See Krishna (2002, Chapters 12-14) and references given therein.

[26] Building on the early contribution by Smith (1966), the analysis of Nautz (1995) studies optimal bidding in divisible-good auctions, where bidders face an exogenous uncertainty about the stop-out rate. As the present analysis shows, however, even with many bidders, the equilibrium distribution of the stop-out rate in repo auctions depends on several factors including the pricing rule, the degree of uncertainty, the remuneration scheme, and the liquidity of the available collateral assets.

[27] The most natural explanation of this fact is asymmetric information in the secondary market, but incorporating this consideration more explicitly would go beyond the scope of the present analysis.

counterparties. A counterparty's demand for (or supply of) liquidity is determined by several factors including reserve requirements, precautionary demand, and idiosyncratic liquidity flows (cf., e.g., Poole, 1968). In fact, in the euro area, precautionary demand plays a subordinated role and is largely dominated by reserve requirements (see, e.g., Bindseil et al., 2003).

Required reserves must be held on average over the so-called reserve maintenance period, which is usually one month. To allow counterparties to collectively satisfy their reserve requirements, the Eurosystem provides liquidity to the banking system through so-called *refinancing operations*. From a volume perspective, the most relevant of those have been the main refinancing operations and the longer-term refinancing operations. Main (longer-term) refinancing operations are conducted on a weekly (monthly) basis to provide liquidity to eligible counterparties over a one-week (three-month) horizon.

In its main refinancing operations, the ECB applies a so-called *neutral allotment policy* which aims at compensating aggregate liquidity imbalances in the banking system (cf. Ewerhart et al., 2008). Deviations from this rule occurred in particular in consequence of tender imbalances, in response to persistent changes of the Eonia spread, and in the context of widespread percussions of liquidity conditions, such as in September 2001 and more recently following August 2007. However, there is no evidence suggesting that the ECB would make the total allotment dependent on incoming bids (cf. Ejerskov et al., 2008).

All central bank lending to the counterparties of the Eurosystem has to be secured by *eligible collateral*. A very broad range of collateral assets is accepted by the Eurosystem, including government bonds, stocks, bank bonds, credit claims, asset-backed securities, and other assets. Should a counterparty fail to transfer collateral, the relevant allotment would not be made, and a penalty would be imposed upon the counterparty. The ECB may decide then to counteract the resulting liquidity imbalance by a so-called fine-tuning operation.

The official documentation describes three basic auction formats. Apart from the *fixed-rate tender*, which is just a proportional rationing scheme, the Eurosystem may use the *variable-rate tender* with either the Dutch (i.e., uniform-price) or the American (discriminatory or pay-your-bid) pricing rule. In the main refinancing operations, the uniform pricing rule has been used only at the very start of Stage III of EMU. Subsequently, the Eurosystem relied upon the fixed-rate tender for nearly one year and a half. In June 2000, however, the fixed-rate tender was abandoned and replaced by the American tender, which has been used since then.

In these discriminatory main refinancing operations, yet not in the longer-term operations, the ECB has chosen to apply a bid floor, the so-called *minimum bid rate*, which has taken over the role of the most prominent policy rate from the rate in the fixed-rate tenders. The minimum bid rate has always been the midpoint of the interest rate corridor formed by the so-called marginal lending and deposit rates. The *marginal rate* (or stop-out rate) is the lowest rate at which allotments are made. The *weighted average rate* is the quantity-weighted average of the interest rate of (the allotted fractions of) successful bids.[28]

The Eurosystem grants *remuneration* of minimum reserve requirements. Specifically, following the completion of a given reserve maintenance period, a counterparty receives an interest payment on its minimum reserve requirement, where the rate applied equals the average of the marginal rates in the main refinancing operations of that maintenance period.

Important secondary market rates include the Eonia, Euribor, Eurepo, and the Eonia swap rate. The euro overnight interest rate *Eonia* is the market index for unsecured overnight interbank lending, computed from the report of panel institutions as a weighted average of contractual rates. The euro interbank offered rate *Euribor* is a maturity bundle of term rates at which a prime bank would be willing to lend euro funds to another prime bank. These rates are computed as an average over a representative panel of prime banks. *Eurepo* is a similar maturity bundle of indices for repurchase agreements involving standard (GC) collateral. The *swap rate* is the market price, with respect to various maturities, for a contract that exchanges fixed against Eonia interest rate payments (cf. Ewerhart et al., 2007).

[28] On an operational level, eligible counterparties may submit up to ten bids, each consisting of a pair of interest rate and quantity. The interest rate in each bid must be a multiple of 1 basis point. There is a similar grid of EUR 0.1 m on the quantity scale, yet this grid is less visible in the data. There is also a minimum amount of EUR 1 m on each interest rate/quantity pair.

Appendix B. Bid curves and linearity

This appendix offers some anecdotal evidence on bid schedules. Fig. 3 shows aggregate bid schedules and supply in four main refinancing operations during the period 2005–2006. The selected examples are typical for a market environment that is characterized by a certain degree of uncertainty. Bid curves for calmer periods look similar, but are less informative because of the interest rate grid. It can be seen that the bid curves share a characteristic shape with a lengthy and relatively flat middle section that is almost a straight line. The examples shown in Fig. 3 therefore suggest that *on an aggregate basis*, the linear equilibrium is largely in line with actual bidding behavior. For additional information on bid schedules, we refer the reader to the empirical literature surveyed in Subsection 2.2.

Appendix C. Mathematical description of the allotment rule

This appendix offers a mathematical description of the allotment rule in the auction model. The description is actually straightforward provided that all participants $i = 1,\ldots,n$ use inverse bid schedules $b_i(.)$ that are strictly declining in quantity. In that case, the allotment to bidder i in the auction will just be stated demand $x_i(r^*)$ at the stop-out rate. However, if the inverse bid schedule of at least one bidder happens to possess a flat section, then there will be a discontinuity in the aggregate bid schedule $x(.)$, so that demand may strictly exceed supply at the stop-out rate. Despite the discontinuity, the situation still remains relatively simple provided that only a single bidder, say bidder i_0, submits an inverse bid schedule with a flat section. In that case, all bidders $i \neq i_0$ will just obtain a regular allotment of $x_i(r^*)$, with rationing being applied exclusively to bidder i_0 who obtains simply the residual supply at the stop-out rate. For our analysis, this is all what may happen in response to a single bidder's deviation from equilibrium behavior involving strictly declining inverse bid schedules.

Thus, all of our results are valid for *any* allotment rule that fully allots bids at rates strictly above the stop-out rate. For completeness, we shall now fully specify the *one* allotment rule with this property that in addition implements proportional rationing at the margin. In doing so, we have to allow for the possibility that more than one bidder submits an inverse bid schedule with a flat section. Two cases must be distinguished. Consider first that case that aggregate bids weakly exceed supply, i.e., that $x(r_{\min}) \geq \tilde{Q}$. Define then $x_i^+(r^*) = \lim_{r \to r^*, r > r^*} x_i(r)$ as bidder i's cumulated bid at an interest rate just above r^*, and let $x^+(r^*) = \sum_{j=1}^{n} x_j^+(r^*)$ denote the corresponding aggregate. In this case, bidder i obtains an allotment

$$q_i^*\left(\tilde{Q}\right) = x_i^+\left(r^*\left(\tilde{Q}\right)\right) + \frac{x_i\left(r^*\left(\tilde{Q}\right)\right) - x_i^+\left(r^*\left(\tilde{Q}\right)\right)}{x\left(r^*\left(\tilde{Q}\right)\right) - x^+\left(r^*\left(\tilde{Q}\right)\right)}\left\{\tilde{Q} - x^+\left(r^*\left(\tilde{Q}\right)\right)\right\}$$

in state $\tilde{Q}$, where the ratio is understood to be one if the denominator vanishes. Thus, if aggregate cumulated bids exceed supply then the allotment is composed of a complete allocation of the part of the bid schedule located strictly above the stop-out rate, and a pro-rata allocation of any flat segment of the bid schedule located at the stop-out rate. In the underbidding case $x(r_{\min}) < \tilde{Q}$, all bids are satisfied, so that the allotment to bidder i amounts to $q_i^*(\tilde{Q}) = x_i(r_{\min})$. The

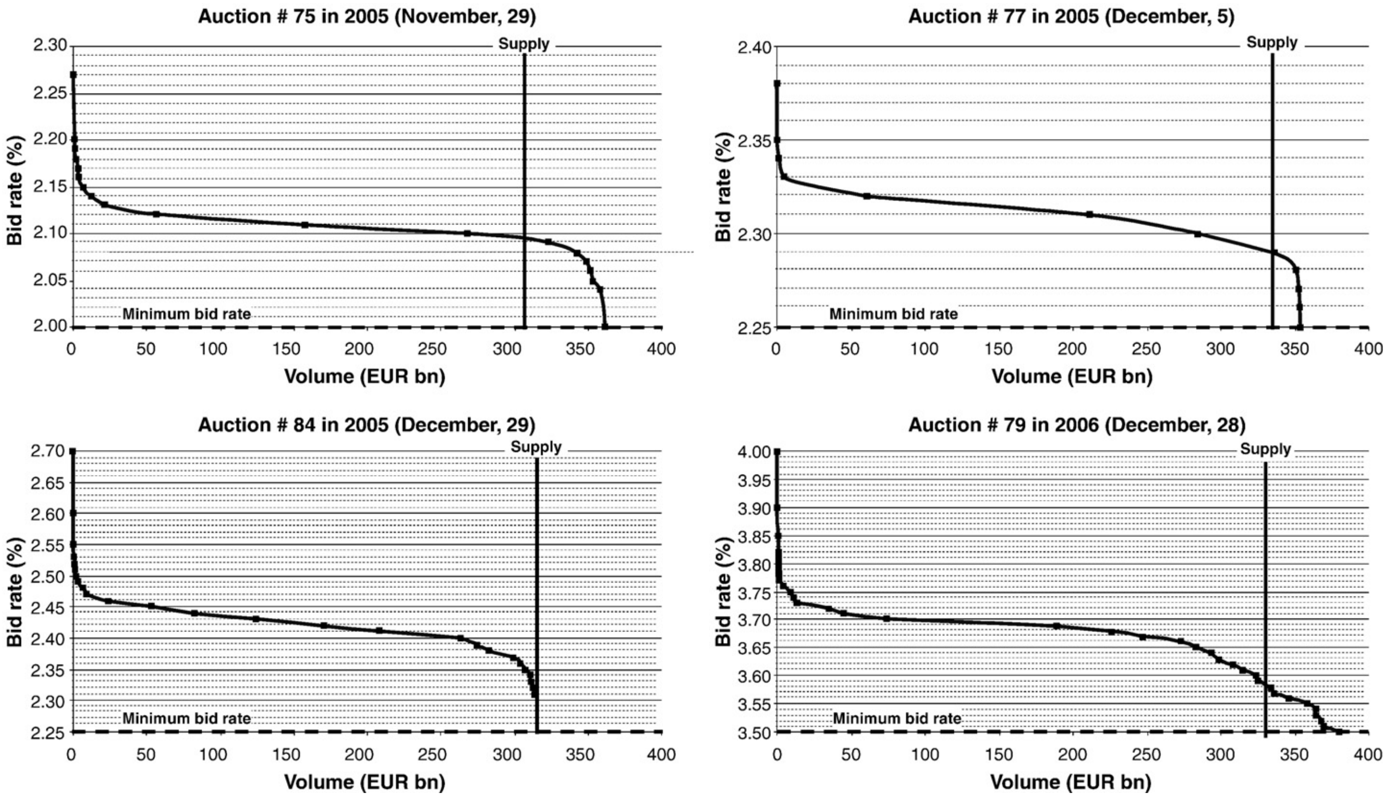

Fig. 3. Aggregate bid schedules.

thereby specified allotment rule applies likewise to the uniform-price and the discriminatory auction.

Appendix D. Proofs

This appendix contains formal proofs of Propositions 1 through 10.

Proof of Proposition 1. Assume that bidder i obtains an allotment q_i^P in the primary market. Then, for $0 \le q_i^P \le q_i^0$, the total net interest paid for funding in the secondary market amounts to

$$C_i\left(q_i^{\mathrm{P}}\right) = r^S\left(q_i^0 - q_i^{\mathrm{P}}\right) + \int_0^{q_i^0 - q_i^{\mathrm{P}}} c_i(q_i)dq_i.$$

Hence, the marginal valuation in the primary market is given by

$$v_i\left(q_i^{\mathrm{P}}\right) = -C_i'\left(q_i^{\mathrm{P}}\right) = r^S + c_i\left(q_i^0 - q_i^{\mathrm{P}}\right) = \overline{v}_i - \frac{q_i^{\mathrm{P}}}{B}.$$

For $q_i^P > q_i^0$, bidder i uses illiquid collateral in the primary market, and offers excess funds $q_i^P - q_i^0$ in the secured market at rate r^S. This proves Eq. (2). □

Proof of Proposition 2. Keep $i \in \{1,\ldots,n\}$ fixed, and assume that each bidder $j \neq i$ submits a schedule $x_j(.)$, where $x_j(r) = B_j^u \max\{\overline{v} - r, 0\}$ for some $B_j^u > 0$. Let $B_{-i}^u = \sum_{j \neq i} B_j^u$. Assume that bidder i uses an admissible bid schedule $x_i(.)$, and consider state $\tilde{Q}$. It will be shown first that *any interest-rate quantity combination* $(r,q_i) = (r^*(\tilde{Q}), q_i^*(\tilde{Q}))$ *resulting from* $(x_1(.),\ldots,x_n(.))$ *in state* $\tilde{Q}$ *satisfies precisely one of the following three conditions*:

(i) $r > \overline{v}$ *and* $q_i = \tilde{Q}$
(ii) $r_{\min} < r \le \overline{v}$ *and* $q_i = \tilde{Q} - (\overline{v} - r)B_{-i}^u \ge 0$
(iii) $r = r_{\min}$ *and* $0 \le q_i \le \tilde{Q} - (\overline{v} - r_{\min})B_{-i}^u$

Clearly, if $x(0) \le \tilde{Q}$, then $r^*(\tilde{Q}) = r_{\min}$ and condition (iii) is satisfied. Assume therefore $x(0) > \tilde{Q}$. Since bidder i is the only bidder with a potentially discontinuous bid schedule,

$$\begin{aligned} q_i^*\left(\tilde{Q}\right) &= x_i^+\left(r^*\left(\tilde{Q}\right)\right) + \tilde{Q} - x^+\left(r^*\left(\tilde{Q}\right)\right) \\ &= \tilde{Q} - \sum_{j \neq i} x_j^+\left(r^*\left(\tilde{Q}\right)\right) \\ &= \tilde{Q} - \sum_{j \neq i} x_j\left(r^*\left(\tilde{Q}\right)\right). \end{aligned}$$

This implies that either (i), (ii), or (iii) will be satisfied, and proves the assertion. Next, we show that *schedule (3) with*

$$B_i^u = \frac{B_i B_{-i}^u}{B_i + B_{-i}^u} \tag{11}$$

is ex-post optimal for bidder i, provided that

$$\overline{Q} \le (\overline{v} - r_{\min})(B_i^u + B_{-i}^u). \tag{12}$$

Fix $\tilde{Q} \in [0;\overline{Q}]$. With the linear specification, bidder i's net interest is given by

$$\Pi_i^u = \int_0^{q_i^*} \{v_i(q_i) - r^*\}dq_i = q_i^*\left\{\overline{v} - r^* - \frac{q_i^*}{2B_i}\right\}, \tag{13}$$

where $r^* = r^*(\tilde{Q})$ and $q_i^* = q_i^*(\tilde{Q})$. Selecting a point (r^*, q_i^*) satisfying (i) obviously cannot be optimal. Moreover, among the points (r,q_i) satisfying condition (iii), the profit-maximizing alternative would entail a quantity $q_i^* = \tilde{Q} - (\overline{v} - r_{\min})B_{-i}^u$. Thus, $q_i^* + (\overline{v} - r^+)B_{-i}^u = \tilde{Q}$ with $r^* \in [r_{\min}; \overline{v}]$. Implicit differentiation delivers $dq_i^*/dr^* = B_{-i}^u$. Using this in the first-order condition resulting from Eq. (13) yields the assertion. Now, we show that *system (11), for* $n\geq3$ *and for* $i=1,\ldots,n$*, has the unique solution*

$$B_i^u = B_i + \frac{B_*^u}{2} - \sqrt{B_i^2 + \left(\frac{B_*^u}{2}\right)^2}, \tag{14}$$

where the parameter B_*^u *is the unique strictly positive root of the equation*

$$\frac{B_*^u}{2} - \frac{1}{n-2}\sum_{i=1}^{n}\left\{\sqrt{B_i^2 + \left(\frac{B_*^u}{2}\right)^2} - B_i\right\} = 0. \tag{15}$$

Define $B_*^u = \sum_{i=1}^n B_i^u$. Using this notation, condition (11) can be rewritten as $(B_i - B_i^u)(B_*^u - B_i^u) = B_i B_i^u$. Solving for B_i^u yields

$$B_i^u = B_i + \frac{B_*^u}{2} \pm \sqrt{\left(B_i + \frac{B_*^u}{2}\right)^2 - B_*^u B_i}.$$

However, only the negative root is economically relevant because otherwise $B_i^u > B_i$, which would contradict Eq. (11). This delivers Eq. (14). Summing up over all $i=1,\ldots,n$, and rearranging yields Eq. (15). To see why Eq. (15) has a unique strictly positive root, note that the equation is certainly satisfied for $B_*^u = 0$. The left-hand side of Eq. (15) has a strictly positive first derivative in $B_*^u = 0$, and is strictly concave for all $B_*^u \geq 0$, so there is at most one root $B_*^u > 0$. Using Eq. (14), this proves uniqueness. On the other hand, for $n\geq3$ fixed, the left-hand side of Eq. (15) follows the asymptotics

$$\frac{B_*^u}{2} - \frac{1}{n-2}\sum_{i=1}^{n}\left\{\sqrt{B_i^2 + \left(\frac{B_*^u}{2}\right)^2} - B_i\right\} \sim -\frac{B_*^u}{n-2} \text{ as } B_*^u \to \infty,$$

and therefore eventually becomes negative for a sufficiently large B_*^u. Invoking the intermediate value theorem proves existence, and thereby the assertion. Clearly, we have $B_i^u < B_i$. Moreover, the right-hand side of Eq. (14) is strictly increasing in B_i, which delivers the monotonicity of equilibrium bids in heterogeneous populations of bidders. The assertion concerning the symmetric case is immediate from Eqs. (14) and (15). □

Proof of Proposition 3. Fix $n\geq3$. Without loss of generality, $B_1(n) \leq B_i(n)$ for all $i=1,\ldots,n$. From the proof of Proposition 2, we know that the slope parameters $B_1^u(n),\ldots,B_n^u(n)$ are characterized uniquely by Eq. (11). Monotonicity implies $B_1^u(n) \leq B_i^u(n)$ for all $i=1,\ldots,n$. In particular, one has $B_{-i}^u(n) \geq (n-1)B_1^u(n)$ for any i. Using Eq. (11), one obtains

$$\frac{B_1^u(n)}{B_1(n)} = 1 - \frac{B_1^u(n)}{B_{-1}^u(n)} \geq \frac{n-2}{n-1}.$$

But then,

$$B_{-i}^u(n) \geq (n-1)B_1^u(n) \geq (n-2)B_1(n) \geq (n-2)\underline{B}.$$

Using Eq. (11) again, one finds

$$\frac{B_i^u(n)}{B_i(n)} \geq 1 - \frac{B_i^u(n)}{(n-2)\underline{B}} \geq 1 - \frac{B_i(n)}{(n-2)\underline{B}} \geq 1 - \frac{\overline{B}}{(n-2)\underline{B}} = 1 - \varepsilon_n,$$

where $\varepsilon_n \to 0$ for $n\to\infty$. But then, for n sufficiently large,

$$\begin{aligned}\overline{Q}(n) &= n\overline{q} \\ &< n(\overline{v} - r_{\min})\frac{\overline{B}}{1-\varepsilon_n} \\ &\leq (\overline{v} - r_{\min})\sum_{i=1}^{n}\frac{B_i(n)}{1-\varepsilon_n} \\ &\leq (\overline{v} - r_{\min})\sum_{i=1}^{n} B_i^u(n),\end{aligned}$$

so that Proposition 2 guarantees the existence of the linear equilibrium. Considering the limit for $n\to\infty$ yields the assertion. □

Proof of Proposition 4. Fix i and assume that bidders $j\neq i$ use a common continuous bid schedule of the form (5) for parameter values $\overline{v}^d$, B^d, and $r_{\min}^d$ satisfying $r_{\min} \leq r_{\min}^d < \overline{v}^d$ and $B^d(\overline{v}^d - r_{\min}^d) = B(\overline{v} - r_{\min}^d)$. We will derive the optimal admissible response $x_i(r)$ of bidder i. Denoting bidder i's inverse bid schedule by $b_i(q_i)$, expected net interest for bidder i is given by

$$E\left(\Pi_i^d\right) = \int_0^{\overline{Q}}\int_0^{q_i^*(\tilde{Q})}(v_i(q_i) - b_i(q_i))dq_i dF(\tilde{Q}),$$

where $F(\tilde{Q}) = 1-(1-(\tilde{Q}/\overline{Q}))^{\alpha}$ denotes the cumulative distribution function of $\tilde{Q}$. Write $Q(q_i) = \{\tilde{Q}\in[0;\overline{Q}] | q_i^*(\tilde{Q}) \geq q_i\}$ for the set of total allotments such that the allotment to bidder i is at least q_i. Then, changing the order of integration yields

$$E\left(\Pi_i^d\right) = \int_0^{\overline{Q}} \mathrm{pr}\{Q(q_i)\}(v_i(q_i) - b_i(q_i))dq_i, \tag{16}$$

where $\mathrm{pr}\{Q(q_i)\} = \int_{Q(q_i)} dF(\tilde{Q})$ is the probability that bidder i receives an allotment of at least q_i. Since the schedules $x_j(.)$ of bidders $j\neq i$ are continuous, and the weakly decreasing inverse bid schedule $b_i(.)$ is necessarily continuous almost everywhere,

$$\mathrm{pr}\{Q(q_i)\} = 1 - F\left(q_i + \sum_{j\neq i} x_j(b_i)\right)$$

for almost every q_i. We drop for the moment the admissibility requirement on $x_i(.)$, and search for a pointwise maximizer $b_i^*(.)$ of the integrand

$$I(b_i, q_i) = \mathrm{pr}\{Q(q_i)\}(v_i(q_i) - b_i) \tag{17}$$

in Eq. (16). Consider first $q_i\in[0;\overline{q}]$. Clearly, $b_i^*(q_i) \leq \overline{v}^d$ because otherwise, lowering $b_i^*(q_i)$ marginally would increase $I(b_i, q_i)$. In the case $r_{\min}^d \leq b_i \leq \overline{v}^d$,

$$\mathrm{pr}\{Q(q_i)\} = \overline{Q}^{-\alpha}\left(\overline{Q} - q_i - (n-1)\left(\overline{v}^d - b_i\right)B^d\right)^{\alpha} \tag{18}$$

for almost any $q_i\in[0;\overline{q}]$. Provided that Eq. (18) holds, the first-order condition has the unique solution

$$b_i^*(q_i) = \frac{\alpha(\overline{v} - q_i/B) + \overline{v}^d}{1+\alpha} - \frac{\overline{Q} - q_i}{(1+\alpha)(n-1)B^d}. \tag{19}$$

Comparing Eq. (19) with $b_i^*(q_i) = \overline{v}^d - q_i/B^d$ yields first Eq. (7) and then Eq. (6). Since $q_i\in[0;\overline{q}]$, this implies $b_i^*(q_i)\in[r_{\min}^d;\overline{v}^d]$, where

$$r_{\min}^d = \overline{v}^d - \frac{\overline{q}}{B^d} = \overline{v} - \frac{\overline{q}}{B} > \overline{v} - \frac{q^0}{B} = r_{\min}$$

is a consequence of Eqs. (1) and (2). Moreover, a straightforward calculation exploiting the first-order condition delivers

$$\frac{\partial^2 I}{\partial b_i^2}\left(b_i^*(q_i), q_i\right) = -(n-1)B^d\frac{1+\alpha}{\alpha}f(q_i + \sum_{j\neq i} x_j(b_i)) < 0.$$

By uniqueness, $b_i^*(q_i)$ is therefore the global maximum on the interval $[r_{\min}^d;\overline{v}^d]$. To complete the characterization of the equilibrium, it remains to be shown that a deviation to a bid $b_i(q_i) < r_{\min}^d$ is not optimal. But in this case

$$I(b_i, q_i) = \left\{\overline{Q} - q_i - (n-1)(\overline{v} - b_i)B\right\}^{\alpha}(v_i(q_i) - b_i),$$

and the first derivative of this expression, for unrestricted b_i, has a unique zero at

$$b_i^{\#}(q_i) = v(q_i) - \frac{\bar{q} - q_i}{(1+\alpha)(n-1)B}.$$

Furthermore, as above, $\partial^2 I/\partial b_i^2(b_i^{\#}(q_i),q_i)<0$. But for $q_i \le \bar{q}$, a straightforward calculation shows that $b_i^{\#}(q_i) \ge r_{\min}^d$ provided that $\alpha \ge 1/(n-1)$. Therefore, by continuity of $I(b_i,q_i)$, lowering $b_i^*(q_i)$ below $r_{\min}^d$ cannot improve bidder i's payoff at $q_i<\bar{q}$. Thus, $b_i^*(q_i)$ is optimal for almost every $q_i \in [0;\bar{q}]$. For $q_i>\bar{q}$, bidder i cannot hope for a positive rent, so we may set $b_i^*(q_i)=\bar{v}-q_i/B$ at these quantities. In sum, $b_i^*(q_i)$ is almost everywhere a pointwise maximizer of (16). Clearly, the schedule $b_i^*(.)$ is then a maximizer of (16). Finally, $b_i^*(.)$ results from the bid schedule $x_i(.)$ given through Eq. (5). □

Proof of Proposition 5. By definition, for any i and any $\tilde{q}>0$, the weighted average tender rate is given by $r_d^{\mathrm{war}}(\tilde{q},n)=\tilde{q}^{-1}\int_0^{\tilde{q}} b_i^*(q_i)dq_i$. Proposition 4 implies $b_i^*(q_i)=\bar{v}^d-q_i/B^d$. Hence,

$$r_d^{\mathrm{war}}(\tilde{q},n) = \bar{v}^d - \frac{\tilde{q}}{2B^d}. \quad (20)$$

Denote by $F_n(.)$ the cumulative distribution function of $\bar{q}=\tilde{Q}/n$. Clearly, $F_n(\tilde{q})=1-(1-(\tilde{q}/\bar{q}))^{\alpha}$, where $\tilde{q}=\bar{Q}/n$. Noting that $E(\tilde{q})=\int_0^{\bar{q}}\tilde{q}dF_n(\tilde{q})=\bar{q}/(1+\alpha)$, a straightforward calculation delivers

$$E(v(\tilde{q}) - r_d^{\mathrm{war}}(\tilde{q},n)) = \frac{\bar{q}}{2B}\frac{\alpha(n+1)}{(1+\alpha)(n+\alpha(n-1))} > 0. \quad (21)$$

Thus, expected underpricing is strictly positive. Also in the limit,

$$\lim_{n\to\infty} E(v(\tilde{q}) - r_d^{\mathrm{war}}(\tilde{q},n)) = \frac{\bar{q}}{2B}\frac{\alpha}{(1+\alpha)^2} > 0. \quad (22)$$

The assertions concerning bid shading and discount follow from the obvious inequalities $r_d^{\mathrm{war}}(\tilde{q},n) \ge r_d^{\mathrm{mar}}(\tilde{q},n)$ and $r_d^{\mathrm{war}}(\tilde{q},n) \ge \bar{q}^{-1}\int_0^{\bar{q}} b(q)dq$, which hold for any $\tilde{q} \in [0;\bar{q}]$ and for any n. □

Proof of Proposition 6. Using Eq. (20), the *average bid level* in the discriminatory auction is given by

$$r_d^{\mathrm{war}}(\bar{q},n) = \bar{v} - \frac{\bar{q}}{2B}\left(1 + \frac{n}{n+\alpha(n-1)}\right),$$

which is strictly increasing in α. We continue with the marginal rate. By Proposition 4, $r_d^{\mathrm{mar}}(\tilde{q},n)=\bar{v}^d-\tilde{q}/B^d$. Taking expectations yields the *expected stop-out rate*

$$E(r_d^{\mathrm{mar}}(\tilde{q},n)) = \bar{v} - \frac{\bar{q}}{B}\frac{n+\alpha(2n-1)}{(1+\alpha)(n+\alpha(n-1))}. \quad (23)$$

Differentiation shows that this term is strictly increasing in α. The *expected average winning rate* is given by

$$E(r_d^{\mathrm{war}}(\tilde{q},n)) = \bar{v} - \frac{\bar{q}}{2B}\frac{2n+\alpha(3n-1)}{(1+\alpha)(n+\alpha(n-1))}, \quad (24)$$

which is likewise strictly increasing in α. It is immediate that the *expected marginal valuation*

$$E(v(\tilde{q})) = \bar{v} - \frac{\bar{q}}{(1+\alpha)B} \quad (25)$$

is strictly increasing in α. Next, one notes that both *expected bid shading*

$$E(v(\tilde{q}) - r_d^{\mathrm{mar}}(\tilde{q},n)) = \frac{\bar{q}}{B}\frac{\alpha n}{(1+\alpha)(n+\alpha(n-1))} \quad (26)$$

and *expected underpricing* (21) are strictly increasing in α for $\alpha < \sqrt{n/(n-1)}$. This condition, however, is safely satisfied if $\alpha<1$, as we assumed. The *expected discount* is given by

$$E(v(\tilde{q})) - \int_0^{\bar{q}} b(q)dq = \frac{\bar{q}}{2B}\frac{\alpha(n+1)+\alpha^2(n-1)}{(1+\alpha)(n+\alpha(n-1))}. \quad (27)$$

Also this expression is strictly increasing in α. Finally, in the limit $n\to\infty$, the explicit expressions derived above imply that strict monotonicity holds in all considered cases. □

Proof of Proposition 7. From Proposition 1, $\bar{v}=r^S+q^0/B$. Plugging this into Eqs. (23), (24), and (25) delivers, after some re-arranging,

$$\left(E(r_d^{\mathrm{mar}}(\tilde{q},n)) - r^S\right)B = \left(q^0 - \bar{q}\right) + \bar{q}\frac{\alpha^2(n-1)}{(1+\alpha)(n+\alpha(n-1))}, \quad (28)$$

$$\left(E(r_d^{\mathrm{war}}(\tilde{q},n)) - r^S\right)B = \left(q^0 - \bar{q}\right) + \bar{q}\frac{\alpha(n+1)+2\alpha^2(n-1)}{2(1+\alpha)(n+\alpha(n-1))},$$

$$\left(E(v(\tilde{q})) - r^S\right)B = \left(q^0 - \bar{q}\right) + \bar{q}\frac{\alpha}{1+\alpha}.$$

Hence, when q^0 and $\bar{q}$ are inflated proportionally, the expected marginal rate, the expected average winning rate, and the expected marginal valuation all increase because of $q^0>\bar{q}$. The claim concerning the standard deviation of the marginal valuation is obvious because $v(\tilde{q})=r^S+(q^0-\tilde{q})/B$. Explicit expressions for the expected bid shading (Eq. (26)), expected underpricing (Eq. (21)), for the expected discount (Eq. (27)), and bid dispersion (Eq. (10)) do all not depend on q^0, yet are strictly increasing in $\bar{q}$. This proves the assertion for finite n. But using the explicit expressions, it is immediate that strict monotonicity holds in all cases also for $n\to\infty$. □

Proof of Proposition 8. With uniform pricing, expected revenues amount to

$$E(\pi^u(\tilde{q},n)) = n\int_0^{\bar{q}} r_u^{\mathrm{war}}(\tilde{q},n)\tilde{q}dF_n(\tilde{q}),$$

where

$$r_u^{\mathrm{war}}(\tilde{q},n) = r_u^{\mathrm{mar}}(\tilde{q},n) = \bar{v} - \frac{\tilde{q}}{B}\frac{n-1}{n-2}. \quad (29)$$

A straightforward calculation exploiting

$$\int_0^{\bar{q}} \tilde{q}^2 dF_n(\tilde{q}) = \frac{2\bar{q}^2}{(1+\alpha)(2+\alpha)}$$

yields that for $n>2$,

$$E(\pi^u(\tilde{q},n)) = \frac{n\bar{q}}{1+\alpha}\left(\bar{v} - \frac{2\bar{q}}{(2+\alpha)B}\frac{n-1}{n-2}\right).$$

With discriminatory pricing, an analogous computation using Eq. (20) delivers

$$E\left(\pi^d(\tilde{q},n)\right) = \frac{n\bar{q}}{1+\alpha}\left(\bar{v} - \frac{\bar{q}}{(2+\alpha)B}\frac{2n+\alpha(2n-1)}{n+\alpha(n-1)}\right) > E(\pi^u(\tilde{q},n)).$$

It is also straightforward to check that

$$\lim_{n\to\infty}\left(E(\pi^d(\tilde{q},n)) - E(\pi^u(\tilde{q},n))\right) = \frac{\bar{q}^2}{(1+\alpha)^2 B} > 0.$$

This proves the assertion. □

Proof of Proposition 9. With remuneration, bidder i's profit in the uniform-price auction amounts to $\prod_i^u + r^*L$, where $\prod_i^u$ is given by Eq. (13). An analysis of the first-order condition shows that in the Dutch variable-rate tender with remuneration, counterparties's bid schedules are identical to those in the basic model, when $\bar{v}$ is replaced by $\bar{v} + \frac{L}{(n-2)B}$. Similarly, in the discriminatory auction with remuneration, bidder i's expected profit is given by

$$E\left(\Pi_i^d + r^*L\right) = \int_0^{\overline{Q}} \int_0^{q_i^*(\tilde{Q})} (v_i(q_i) - b_i(q_i))dq_i + r^*(\tilde{Q})LdF(\tilde{Q}).$$

Assuming that bank $j \neq i$ bids as predicted in Proposition 4, one finds

$$r^*(\tilde{Q}) = \bar{v}^d - \frac{\tilde{Q} - q_i^*(\tilde{Q})}{(n-1)B^d}.$$

Hence, up to a constant, bidder i's profit is given by

$$E\left(\Pi_i^d + r^*L\right) = const. + \int_0^{\overline{Q}} \int_0^{q_i^*(\tilde{Q})} \left(v_i(q_i) + \frac{L}{(n-1)B^d} - b_i(q_i)\right)dq_i dF\left(\tilde{Q}\right).$$

Thus, if $\bar{v}$ is replaced by $\bar{v} + \frac{L}{B}\frac{\alpha}{n+\alpha(n-1)}$ in the basic model, one obtains an equilibrium for the American tender with remuneration. This proves the existence of the claimed equilibrium in the auctions with remuneration. We consider now first the case without remuneration, i.e., $L = 0$. Taking expectations in Eq. (29) yields

$$E(r_u^{\text{mar}}(\tilde{q}, n)) = \int_0^{\overline{q}} r_u^{\text{mar}}(\tilde{q}, n)dF_n(\tilde{q}) = \bar{v} - \frac{\overline{q}}{(1+\alpha)B}\frac{n-1}{n-2}.$$

A straightforward calculation using Eq. (28) shows that

$$E(r_u^{\text{mar}}(\tilde{q}, n)) - E(r_d^{\text{mar}}(\tilde{q}, n)) = \frac{\overline{q}}{(1+\alpha)B}\left(\frac{\alpha n}{n+\alpha(n-1)} - \frac{1}{n-2}\right).$$

Thus, $E(r_u^{\text{mar}}(\tilde{q},n)) < E(r_d^{\text{mar}}(\tilde{q},n))$ provided that $\alpha > \alpha^* = n/(n^2-3n+1)$. For $n \geq 4$, we have $\alpha^* < 1$. For α fixed and $n \to \infty$,

$$\lim_{n\to\infty} E(r_u^{\text{mar}}(\tilde{q}, n)) - E(r_d^{\text{mar}}(\tilde{q}, n)) = \frac{\alpha\overline{q}}{(1+\alpha)^2 B} > 0.$$

The corresponding claims for the case with remuneration follow from

$$\frac{L}{(n-2)B} \geq \frac{L}{B}\frac{\alpha}{n+\alpha(n-1)},$$

i.e., from the fact that the strategic upwards drift on the expected marginal rate is stronger for the uniform-price auction than for the discriminatory auction. □

Proof of Proposition 10. By returning to the explicit expressions derived in the proof of Proposition 7, it is easy to check that a ceteris paribus decrease of B to B' is throughout equivalent to a simultaneous inflation of q^0 and $\overline{q}$ by the factor $B/B' > 1$. □

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